$x_i^{(0)} = 0$, $i = 1$–4, and also $x_i^{(5)} = 0$, $i = 1, 2, 3, 4$. The only unknown is $x_1^{(1)}$. (It is to be noted that the number of unknowns is only one irrespective of the number of stages unlike in other available methods.) To determine $x_1^{(1)}$, unidimensional search procedure was used. The results are given in Table 2. Using the values of interacting forces given in Table 2, the internal forces in the members can be calculated.

Table 1 Properties of Elements

	Horizontal members	Vertical members	Diagonal members
1) Area of cross section, sq. in.	00.25	00.20	00.20
2) Length of the member, in.	15.00	20.00	25.00
3) Young's modulus in ksi	10^4	10^4	10^4
4) Characteristic force F_0, kips	11.65	08.54	08.54
5) Characteristic force S_0, kips	10.13	08.10	08.10
6) Parameter n_1	05.00	05.00	05.00
7) Parameter n_2	07.00	07.00	07.00

Table 2 State variables for the example problem

	$X^{(1)}$, kips	$X^{(2)}$, kips	$X^{(3)}$, kips	$X^{(4)}$, kips	$X^{(5)}$, kips
1)	−2.2827	−1.5534	0.9148	3.7984	−0.0000
2)	+3.4673	+1.1966	−1.3352	−2.4516	0.0000
3)	0.0000	4.3125	6.3750	4.6875	0.0000
4)	0.0000	4.3125	6.3750	4.6875	0.0000

Conclusions

A method of analysing trusses made of nonlinear elastic materials was presented. Considerable reduction in the number of unknowns to be determined is possible for structures which can be partitioned into a series of substructures with a consequent reduction in computer storage capacity requirements. Possible disadvantages with the method may be that the round-off errors may tend to be multiplied as the calculations are carried out from one stage to next stage.

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Annular Plate with Supporting Edge Beams

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Introduction

Annular plates are used extensively in some aerospace structures. Here, an annular plate framing into simply supported edge beams and loaded by a concentrated load is considered as shown in Fig. 1. The deflection of the plate, w, is given within the theory of thin elastic plates by the expression

$$\nabla^4 w = q/D \tag{1}$$

where ∇^2 is the Laplacian operator applied twice, q is the intensity of the lateral loading, and D is the flexural rigidity of the plate.

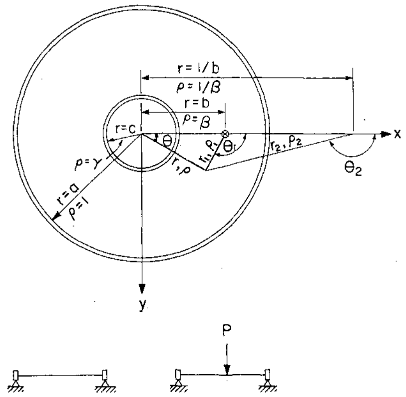

Fig. 1 Annular plate with supporting edge beams.

Analysis

The general solution of Eq. (1) is expressed as

$$w = w_0 + w_1 \tag{2}$$

where w_0 is the particular solution and w_1 the homogeneous solution. The particular solution for the annular plate problem subjected to a concentrated force is taken in the form[1]

$$w_0 = (\rho a^2/8\pi D)\rho_1^2 \log(\rho_1/\beta\rho_2) \tag{3}$$

where the radial coordinate r is nondimensionalized by setting

$$\rho = r/a \tag{4}$$

Equation (3) represents the proper singularity term which must be present in the solution due to the concentrated force.[2]

The homogeneous solution is[3]

$$w_1 = \frac{\rho a^2}{8\pi D}\sum_{n=0}^{\infty} R_n \cos n\theta \tag{5}$$

where

$$R_0 = A_0 + B_0\rho^2 + C_0 \log\rho + D_0\rho^2 \log\rho \tag{6}$$

$$R_1 = A_1\rho + B_1\rho^3 + C_1\rho^{-1} + D_1\rho\log\rho \tag{7}$$

$$R_n = A_n\rho^n + B_n\rho^{-n} + C_n\rho^{n+2} + D_n\rho^{-n+2} \quad n \geq 2 \tag{8}$$

where A, B, C, and D are constants.

The deflection function is required to satisfy the following boundary conditions[4]

$$w = 0 \quad \text{on } \rho = 1 \tag{9}$$

$$w = 0 \quad \text{on } \rho = \gamma \tag{10}$$

$$M_\rho = (D/a^2)(\partial/\partial\rho)[\kappa_1 w - \lambda_1(\partial^2 w/\partial\theta^2)] \quad \text{on } \rho = 1 \tag{11}$$

$$M_\rho = -(D/a^2)(\partial/\partial\rho)[\kappa_2 w - \lambda_2(\partial^2 w/\partial\theta^2)] \quad \text{on } \rho = \gamma \tag{12}$$

Here,

$$\kappa_1 = K_1/aD; \quad \lambda_1 = L_1/aD \tag{13}$$

$$\kappa_2 = K_2/\gamma aD; \quad \lambda_2 = L_2/\gamma aD \tag{14}$$

where K and L are the flexural and torsional rigidities of the edge

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beams with subscript 1 applying to the outside and subscript 2 to the inside edge beam.

On $\rho = 1$, w_0 satisfies Eq. (9) identically. This is not the case on $\rho = \gamma$, however, and expansion of w_0 into a series of cosine multiples of θ, valid in the region $\gamma \leq \rho \leq \beta$, is necessary. This yields

$$w_0 = \frac{\rho a^2}{8\pi D} \sum_{n=0}^{\infty} (\phi_n \rho^2 + \psi_n)\rho^n \cos n\theta \tag{15}$$

where

$$\phi_0 = 1 + \log\beta - \beta^2 \tag{16a}$$

$$\psi_0 = \beta^2 \log\beta \tag{16b}$$

$$\phi_1 = \beta - 1/2\beta - \beta^3/2 \tag{16c}$$

$$\psi_1 = \beta^3 - \beta - 2\beta\log\beta \tag{16d}$$

$$\phi_n = (1/n)(\beta^n - 1/\beta^n) - [\beta/(n+1)](\beta^{n+1} - 1/\beta^{n+1}) \tag{16e}$$

$$\psi_n = (\beta^2/n)(\beta^n - 1/\beta^n) - [\beta/(n-1)](\beta^{n-1} - 1/\beta^{n-1}) \tag{16f}$$

The boundary conditions may now be applied to determine the unknown coefficients yielding

$$A_0 + B_0 = 0 \tag{17a}$$

$$A_0 + B_0\gamma^2 + C_0\log\gamma + D_0\gamma^2\log\gamma = -\phi_0\gamma^2 - \psi_0 \tag{17b}$$

$$2B_0(\kappa_1 + \nu + 1) + C_0(\kappa_1 + \nu - 1) + D_0(\kappa_1 + \nu + 3) = (\beta^2 - 1)(\kappa_1 + \nu + 3) \tag{17c}$$

$$2B_0(\kappa_2 - \nu - 1) + C_0(\kappa_2 - \nu + 1)\gamma^{-2} + D_0[(2\log\gamma + 1)(\kappa_2 - \nu - 1) - 2] = 2\phi_0(-\kappa_2 + \nu + 1) \tag{17d}$$

$$A_1 + B_1 + C_1 = 0 \tag{18a}$$

$$A_1\gamma + B_1\gamma^3 + C_1\gamma^{-1} + D_1\gamma\log\gamma = -\phi_1\gamma^3 - \psi_1\gamma \tag{18b}$$

$$A_1(\kappa_1 + \lambda_1) + B_1[3(\kappa_1 + \lambda_1) + 2(3 + \nu)] - C_1[\kappa_1 + \lambda_1 - 2(1 - \nu)] + D_1(\kappa_1 + \lambda_1 + \nu + 1) = -4\beta(1 - \beta^2) \tag{18c}$$

$$A_1(\kappa_2 + \lambda_2)\gamma^{-1} + B_1[3(\kappa_2 + \lambda_2) - 2(3 + \nu)]\gamma + C_1[-\kappa_2 - \lambda_2 - 2(1 - \nu)]\gamma^{-3} + D_1[(\log\gamma + 1)(\kappa_2 + \lambda_2) - \nu - 1]\gamma^{-1} = (-3\phi_1\gamma - \psi_1/\gamma)(\kappa_2 + \lambda_2) + 2\phi_1\gamma(3 + \nu) \tag{18d}$$

and, for $n \geq 2$

$$A_n + B_n + C_n + D_n = 0 \tag{19a}$$

$$A_n\gamma^n + B_n\gamma^{-n} + C_n\gamma^{n+2} + D_n\gamma^{-n+2} = -\phi_n\gamma^{n+2} - \psi_n\gamma^n \tag{19b}$$

$$A_n[n(\kappa_1 + n^2\lambda_1) - n(n-1)(\nu - 1)] + B_n[-n(\kappa_1 + n^2\lambda_1) - n(n+1)(\nu - 1)] + C_n[(n+2)(\kappa_1 + n^2\lambda_1) + (n+2)(n+1) + \nu(n+2) - \nu n^2] + D_n[(2-n)(\kappa_1 + n^2\lambda_1) + (n-2)(n-1) + \nu(2-n) - \nu n^2] = -4\beta^n(1 - \beta^2) \tag{19c}$$

$$A_n[(\kappa_2 + n^2\lambda_2)n + n(n-1)(\nu-1)]\gamma^{n-2} + B_n[-n(\kappa_2 + n^2\lambda_2) + n(n+1)(\nu-1)]\gamma^{-n-2} + C_n[(n+2)(\kappa_2 + n^2\lambda_2) - (n+2)(n+1) - \nu(n+2) + \nu n^2]\gamma^n + D_n[(2-n)(\kappa_2 + n^2\lambda_2) - (n-2)(n-1) - \nu(2-n) + \nu n^2]\gamma^{-n} = \gamma^n\phi_n[(n+2)(n+1) + \nu(n+2) - \nu n^2] + \gamma^{n-2}[n(n-1) + \nu n - \nu n^2]\psi_n - \gamma^n\phi_n(n+2)(\kappa_2 + n^2\lambda_2) - n\psi_n\psi^{n-2}(\kappa_2 + n^2\lambda_2) \tag{19d}$$

from which all the unknowns can be determined.

Results and Conclusions

In order to interpret the results obtained, the deflection $w(\rho)$ along the radial line $\theta = 0$ was plotted for a plate with $\gamma = 0.2$, $\beta = 0.5$, and $\kappa_i = \lambda_i$. Deflections have also been plotted along the circumference of a circle of $\rho = 0.55$. These results are shown in Figs. 2 and 3, respectively.

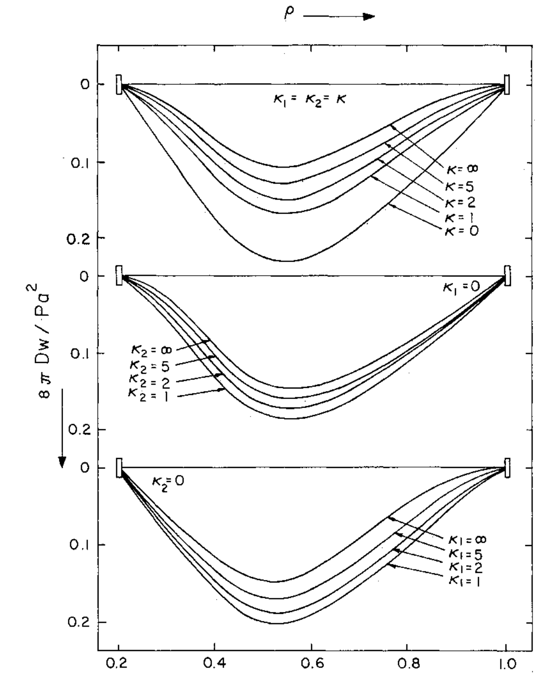

Fig. 2 Deflection under concentrated load.

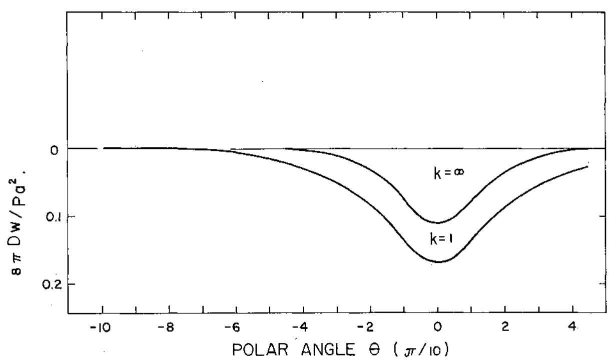

Fig. 3 Variation of deflection with polar angle.

It is clear from the results presented that the deflection of the plate decreases as the stiffness of the edge beams increases. The deflection of a simply supported plate may be decreased considerably by providing edge beams of relatively low stiffness. Thus, the difference in deflection between a plate with edge beams of $\kappa = 5$ and $\kappa = \infty$ is small compared to the difference in deflections between plates of $\kappa = 0$ and $\kappa = 1$. The limiting values of $\kappa = 0$ and $\kappa = \infty$ represent the cases of the simply supported and clamped plate, respectively.

Plate deflections are comparatively large in the immediate vicinity of the point of application of the concentrated load but diminish rapidly as the distance from that point increases. In the region $\pi/2 \leq \theta \leq 3\pi/2$, deflections are seen to be very small. This is particularly true in the case of the clamped plate ($\kappa = \infty$), where, for all practical purposes, the plate may be considered as clamped along the lines $\theta = \pm\pi/2$.

The magnitude of the inside radius of the plate directly affects the deflection. For a plate with $\gamma = 0.6$, clamping will reduce the deflection to about $\frac{1}{3}$ that of a simply supported plate, but for a plate of $\gamma = 0.2$, clamping will only reduce the deflection to about $\frac{1}{2}$ that of the same plate simply supported.

Finally, it was found that the torsional rigidity of the edge beams has very little effect on the deflection of the plate. This is particularly true of beams with high-flexural rigidity.

A system of equations analogous to Eqs. (17–19) may be derived for other loading conditions such as the case of a hydrostatic or uniform load. Most situations arising in practice may then be solved by superposition. In general, since the solution presented herein is the Green's function for this plate system, the general expression the plate deflection under any distributed loading is given by the expression

$$w_q = \int_0^{2\pi} \int_\gamma^1 \frac{q(r, \theta)}{P} w\rho \, d\rho \, d\theta \tag{20}$$

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Thermoelastic Stress in a Rod due to Distributed Time-Dependent Heat Sources

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Introduction

PROBLEMS on determination of thermo-elastic stress and displacement in thin rods, finite or semi-infinite, under various mechanical and thermal boundary conditions have been considered by many authors, Sneddon,[1] Das,[2] Roy Choudhuri.[3] Recently moving heat source moving with a constant velocity along a finite rod has been considered by Roy Choudhuri[4] (1971). In this Note, a simple problem of thermal stress and displacement in a thin finite rod has been considered when the heat sources continuously distributed over a finite portion of the rod vary with time according to the ramp-type function, and when one end of the rod is fixed with the other free, both the ends being kept at zero temperatures. Laplace transform has been found convenient for the solution of the problem. It is believed that this particular problem has not been, so far, considered by any author.

Formulation of the Problem: Governing Equations

We consider a thin elastic rod of length l occupying the region $D: 0 \le x \le l$. The rod is heated due to heat sources which vary with time according to the ramp-type function and are distributed continuously over the length $x_0 < x < x_1$. The ends of the rod are kept at zero temperatures. If $T = T(x, t)$ is the excess of temperature over T_0, the absolute temperature of the rod in a state of zero stress and strain, then normal stress $\sigma = \sigma(x, t)$ is connected with u and T by the relation

$$\sigma = E(\partial u/\partial x - \alpha T) \tag{1}$$

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where E is Young's modulus, α is the linear coefficient of thermal expansion and $u = u(x, t)$ is the displacement at x and at time t.

In the absence of body forces, equation of motion becomes

$$\partial\sigma/\partial x = \rho(\partial^2 u/\partial t^2) \tag{2}$$

where ρ = density of the material of the rod. Eliminating σ and u between Eqs. (1) and (2), we obtain the wave equations satisfied by displacement and stress as

$$\partial^2 u/\partial x^2 = (1/v^2)(\partial^2 u/\partial t^2) + \alpha(\partial T/\partial x) \tag{3}$$

$$\partial^2 \sigma/\partial x^2 = (1/v^2)(\partial^2 \sigma/\partial t^2) + \rho\alpha(\partial^2 T/\partial t^2) \tag{4}$$

where $v = (E/\rho)^{\frac{1}{2}}$ is the velocity of elastic wave propagation.

Initial and Boundary Conditions

We suppose that the system was at rest initially. Thus the initial conditions are

$$u = \partial u/\partial t = 0, \quad \sigma = \partial\sigma/\partial t = 0 \quad \text{for} \quad 0 \le x \le l, \quad t = 0$$

and $T = 0$ for $0 \le x \le l$, $t = 0$. Also $T = 0$ at both $x = 0, l$. Since one end is kept fixed and the other free, $u = 0$ at $x = 0$, and $\partial u/\partial x = 0$ at $x = l$. The last condition follows from Eq. (1).

Solution of the Heat-Conduction Equation

Since the heat sources are continuously distributed over $x_0 < x < x_1$ and vary with time according to the ramp-type function, heat conduction equation is

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{k}\frac{\partial T}{\partial t} = -\frac{Q(x, t)}{k} = -\frac{Q_0}{k}\{H(x - x_0) - H(x - x_1)\}f(t) \tag{5}$$

where k = thermal diffusivity, $Q = w/\rho c$ where w = quantity of heat generated by the heat sources per unit time and volume, c = specific heat and

$$f(t) = (f_0/t_0)t \quad \text{for} \quad 0 \le t \le t_0$$
$$= f_0 \quad \text{for} \quad t \ge t_0$$

$H(x)$ = Heaviside step function, f_0 = const. The boundary and initial conditions for temperature are

$$T(0, t) = T(l, t) = 0 \quad \text{and} \quad T(x, 0) = 0.$$

Introducing dimensionless quantities $\xi = x/l$, $\tau = kt/l^2$, $\Theta = T/T_0$, Eq. (5) reduces to

$$(\partial^2\theta/\partial\xi^2) - (\partial\theta/\partial\tau) = -q_0 F(\tau)\{H(\xi - \xi_0) - H(\xi - \xi_1)\} \tag{6}$$

where $q_0 = Q_0 l^2/kT_0$, $\xi_0 = x_0/l$, $\xi_1 = x_1/l$,

$$F(\tau) = \frac{f_0}{\tau_0}\tau \quad \text{for} \quad 0 \le \tau \le \tau_0 \tag{7}$$
$$= f_0 \quad \text{for} \quad \tau \ge \tau_0$$

and $\tau_0 = kt_0/l^2$. Taking Laplace transform of (6), we obtain,

$$\frac{d^2\bar\theta}{d\xi^2} - p\bar\theta = -\frac{q_0 f_0}{\tau_0 p^2}(1 - e^{-\tau_0 p})\{H(\xi - \xi_0) - H(\xi - \xi_1)\} \tag{8}$$

with

$$\bar\theta(0, p) = \bar\theta(l, p) = 0. \tag{9}$$

As a solution of (8) satisfying (9), we assume

$$\bar\theta(\xi, p) = \sum_{n=1}^{\infty} A_n \sin(n\pi\xi) \tag{10}$$

where A_n is independent of ξ. We assume

$$H(\xi - \xi_0) - H(\xi - \xi_1) = \sum_{n=1}^{\infty} B_n \sin(n\pi\xi) \tag{11}$$